An Open Singularity-Free Cosmological Model with Inflation

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Abstract

In the light of recent observations which point to an open universe ($\Omega_0 < 1$), we reconsider the singularity-free models, originally constructed for closed universes. Our model starts from a nonsingular state called prematter, governed by an inflationary equation of state $P = (\gamma_p - 1)\,\rho$, where γ_p is a small positive parameter representing the initial vacuum dominance of the universe. Unlike the closed models universe cannot be initially static hence, starts with an initial expansion rate represented by the initial value of the Hubble constant $H(0)$. Therefore, our model is a two-parameter universe model $(\gamma_p, H(0))$. During the prematter phase, due to the unusual characteristic of the equation of state, universe heats up even though it expands. When the temperature in the universe reaches the Planck temperature (T_{pl}) which is taken as the maximum attainable physical temperature, a first order phase transition carries the universe into the radiation era. Then the universe starts to behave as predicted in the standard model. The model proposed in this work predicts a value between 60 and 80 $Km \cdot \sec^{-1} \cdot Mpc^{-1}$ for the present value of the Hubble constant (H_0) and the predicted value of Ω_0 lies between 0.3 and 0.6. Comparing the predictions of this model for the present properties of the universe with the recent observational results, we argue that the model constructed in this paper could be used as a realistic universe model.

I. INTRODUCTION

Cosmological models constructed by using the Einstein's field equations and known as the "standard model" describes the universe as a homogenous and isotropic medium starting from an initial singularity and filled with a perfect fluid characterized by an equation of state. According to the standard model the universe is mainly represented by two eras called "radiation" and "matter", which are connected by a first order phase transition. During the radiation era the universe is filled with isotropic blackbody radiation with an equation of state $P = \rho/3$. After a phase transition, radiation is decoupled from the matter and the universe becomes matter dominated as we observe today. During this era the universe is assumed to be composed of incoherent matter exerting zero pressure and thus having an equation of state given as $P = 0$. Considering the small relative motions of the galaxies in the universe and the large intergalactic distances, this assumption is reasonable. Within the framework of the standard model the universe starts its journey as a cosmic fireball emerging from an initial singularity known also as the "big bang". During its entire history it cools down due to its expansion. As a sign of this cooling we now observe a background radiation at 2.7 K that fills our universe.

Although the standard cosmological model has been successful in explaining the homogenous expansion of the universe and the 2.7 K cosmic microwave background radiation, it has some shortcomings like the initial singularity, horizon (or causality), flatness, homogeneity and isotropy problems [1,2]. Among these problems the initial singularity problem may be the one which weakens the model much more than the others, in the sense that it causes infinities in the physical quantities such as the density, pressure and temperature. Since these infinities cannot be accepted by any model which claims to be physical, during the past two decades, several authors have considered the possibility of describing the universe with a singularity-free cosmological model [3-6].

Rosen [3] suggested that a cosmological model could be constructed by starting the evolution of the universe from a limiting density called the Planck density (ρ_{pl}) which could be constructed by using the universal constants: the speed of light (c), Planck's constant ($\hbar$), and the gravitational constant (G): $\rho_{pl} = c^5/G^2\hbar = 5.1566 \cdot 10^{93} gr/cm^3$ [7].

In a subsequent paper Israelit and Rosen (hereafter IR) constructed a nonsingular closed universe model [4]. According to this model the universe starts from a cold nonsingular Planck-state and undergoes a rapid expansion during which it heats. This period of rapid expansion is characterized by a vacuum equation of state in the form $P = -\rho$ [8,9], and because of the fact that matter would be under extreme conditions during this period and behaves very differently from the ordinary matter, it is called "prematter". After this period of rapid expansion, known also as inflation, there is a period of transition into the radiation era. Once the universe enters the radiation era it behaves as predicted in the standard model. In this model all different eras and transitions between these eras are governed by suitably chosen equation of states which reduce under appropriate conditions to that of the desired era.

In the work done by Starkovich and Cooperstock (hereafter SC) [5] a cosmological field theory was proposed to describe the evolution of the universe by means of a single scalar field which describes all the phases that the universe undergoes. The cosmological model built upon the considerations of this field theory describes an oscillating singularity-free closed universe in which the transitions between eras (prematter, radiation, matter) are the results of thermodynamically imposed boundary conditions rather than a finely tuned mechanism which is undesirable for any cosmological theory. In this model, the universe initially is in a "vacuum-like" state ($P \simeq -\rho$) and inflation arises due to this "vacuum-like" characteristic of the equation of state.

In the paper given by Bayin-Cooperstock-Faraoni (hereafter BCF) [6], a closed singularity-free cosmological model was built upon the ideas presented in the paper by S.C. [5]. Furthermore, the form of the potential which is responsible for the inflation was derived unlike most other approaches which assume the form of the scalar field potential a priori.

In all the models mentioned above the universe is modelled as a closed ($k = 1$) Robertson-Walker (RW) space-time. Although they produce realistic results about the present properties of the universe such as Hubble constant, age and density, they have to be reconsidered in the light of recent observational data [10-16]. Recent observations predict values between 60 and 80 $Km \cdot s^{-1} \cdot Mpc^{-1}$ for the present value of Hubble constant (H_0). On the other hand, IR found a value of $\simeq 46$ $Km \cdot s^{-1} \cdot Mpc^{-1}$ and SC estimates values between 33 and 44 $Km \cdot s^{-1} \cdot Mpc^{-1}$. Unlike mentioned in the paper given by BCF, the model that they constructed is sensitive to the temperature of the last phase transition T_m. In their model the present value of the Hubble constant is subject to an upper limit which is determined by the value of T_m. BCF used a value of $2.4805 \cdot 10^4 K$ for T_m and obtained an upper limit of 46.7 $Km \cdot s^{-1} \cdot Mpc^{-1}$ for the present value of the Hubble constant.

Recent observations also point out that the universe has an $\Omega_0 (\Omega \equiv \rho/\rho_c \ , \ \rho_c \equiv 3H^2/8\pi)$ value lower than unity which means that the universe is spatially open [17-21]. In the light of these we propose an open singularity-free cosmological model with the same motivations as in BCF. However, initially static condition ($\dot{a}(0) = 0$) used by the closed universe models can no longer be used in the open universe case since it leads to negative energy densities. For this reason in our model universe starts with an initial velocity. Hence, we have a two-parameter universe model in which one of the parameters characterizes the equation of state used to describe the prematter era, and the other one corresponds to the initial expansion rate of the universe.

The remainder of this paper is organized as follows. In Sec. II, we give a description of the cosmological model with its dynamical equations and solve them analytically under the initial conditions inspired by the Gliner's ideas and the thermodynamically imposed boundary conditions. We also investigate different regimes corresponding to different values of the parameters and we discuss numerical results for these regimes. We present our discussions and conclusions in Sec. III.

II. DESCRIPTION OF THE MODEL

A. Field Equations

Our model describes an open, spatially homogenous and isotropic universe with a space-time geometry given by the RW line element:

$$ds^2 = dt^2 - a^2(t) \left[\frac{dr^2}{1 + r^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right], \tag{1}$$

where (t, r, θ, ϕ) are the comoving coordinates, $a(t)$ is the scale factor which represents the size of the universe.

The dynamics of our model universe is governed by the Einstein's gravitational field equations given as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu}. \tag{2}$$

For the RW metric, field Eqs. (2) lead us to the following differential equations:

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2} = \frac{8\pi}{3}\rho, \tag{3}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2} = -8\pi P, \tag{4}$$

where ρ and P are the energy density and pressure in the universe, where we have used perfect fluid energy momentum tensor in comoving coordinates. Here a dot denotes differentiation with respect to the cosmic time t and we use the units so that the gravitational constant $M_{pl} = G^{-1/2} = 1$, where M_{pl} is the Planck mass.

Combining Eqs. (3) and (4) and using an equation of state given as

$$P = (\gamma - 1)\,\rho, \tag{5}$$

one obtains the following relation:

$$\frac{\ddot{a}}{a} + 4\pi \left(\gamma - \frac{2}{3}\right)\rho = 0. \tag{6}$$

Where, γ is assumed to be a constant parameter during each era in the history of the universe.

We can eliminate ρ between Eqs. (3) and (6) and obtain an equation involving only the scale factor $a(t)$:

$$\left(\frac{\ddot{a}}{a}\right) + \left(\frac{3}{2}\gamma - 1\right)\left(\frac{\dot{a}^2 - 1}{a^2}\right) = 0. \tag{7}$$

To solve this equation for any γ it is advantageous to work in conformal time η which is defined as

$$dt = a\,(\eta)\,d\eta. \tag{8}$$

Using this transformation, Eqs. (3), (6) and (7) can now be written as

$$\left(\frac{a'}{a^2}\right)^2 - \frac{1}{a^2} = \frac{8\pi}{3}\rho, \tag{9}$$

$$\frac{a''}{a^3} - \left(\frac{a'}{a^2}\right)^2 + 4\pi\left(\gamma - \frac{2}{3}\right)\rho = 0, \tag{10}$$

$$\frac{a''}{a} + \left(\frac{3}{2}\gamma - 2\right)\left(\frac{a'}{a}\right)^2 - \left(\frac{3}{2}\gamma - 1\right) = 0, \tag{11}$$

where a prime denotes differentiation with respect to the conformal time. If we define a new dependent variable given as

$$u \equiv \frac{a'}{a} = \frac{d\ln a}{d\eta}, \tag{12}$$

Eq. (11) becomes the Riccati equation of the following form

$$u' + cu^2 - c = 0, \tag{13}$$

where

$$c = \frac{3}{2}\gamma - 1. \tag{14}$$

We consider equation of states such that $c \neq 0$. We could solve Eq. (13) by defining a new variable u,

$$u \equiv \frac{1}{c}\frac{w'}{w} = \left[\ln\left(w^{1/c}\right)\right]'. \tag{15}$$

In terms of this new variable Eq. (13) could be written as

$$w'' - c^2 w = 0, \tag{16}$$

which has two linearly independent solutions given as

$$a(\eta) = a_0 \left[\sinh\left(\eta c + \delta\right)\right]^{1/c}, \tag{17}$$

$$a(\eta) = a_0 \left[\cosh\left(\eta c + \delta\right)\right]^{1/c}, \tag{18}$$

where a_0 and δ are the integration constants.

These solutions correspond to different signs of the energy density ρ. Assuming that the universe is filled with positive energy, we eliminate Eq. (18) and consider only Eq. (17) as the solution for the scale factor of the universe.

In this work, we model the universe as a series of perfect fluid eras connected by first order phase transitions. According to the considerations of this scenario, the universe starts with a period of rapid expansion called inflationary era. This period is characterized by an equation of state in the form $P \simeq -\rho$ and from Eq. (6) it could be seen that inflation arises due to this vacuum-like characteristic of the equation of state. Since matter would be under extreme conditions during this period and behaves very differently from the ordinary matter, it is called "prematter" [4]. During this period due to the unusual characteristic of the equation of state, temperature increases although the universe expands enormously. We assume that inflation continues until the temperature reaches the maximum allowed temperature i.e. the Planck temperature $T_{pl} = 1.4169 \cdot 10^{32}\ K$. This behavior, which does not necessitate a "re-heating mechanism" as in the other models of the universe, follows from the vacuum like characteristic of the equation of state used to describe the universe in this era.

Once the inflationary era ends, the equation of state changes discontinuously into that of the isotropic radiation and the universe starts cooling down. From there on, our description of the universe is like that of the standard model. Therefore, there are basically three eras in our model and two transition periods linking these eras into each other by first order phase transitions. We could summarize our model as

a) **The inflationary (prematter) era** $(0 \leq \eta \leq \eta_r)$**:** During this era, the equation of state is given by $P = (\gamma - 1)\rho$ with $\gamma = \gamma_p \sim 10^{-3}$. The case $\gamma_p = 0$ is excluded because it gives rise to eternal inflation. Indeed, γ_p is a parameter which determines how close the universe is to the vacuum equation of state: $P = -\rho$ [8,9].

b) **The radiation era** $(\eta_r \leq \eta \leq \eta_m)$**:** During this era, $\gamma = \gamma_r = 4/3$, and the equation of state takes the form of that of the isotropic radiation: $P = \rho/3$.

c) **The matter era** ($\eta_m \leq \eta$)**:** During this era, $\gamma = \gamma_m = 1$, and the corresponding equation of state is the dust equation of state, which is given as $P = 0$.

We have used p, r and m as subscripts (or superscripts) to denote the prematter, radiation and matter eras, respectively.

Before discussing the initial and the boundary conditions for the dynamics of the universe, it would be convenient to derive an expression governing the time evolution of the temperature T. First, we consider the first law of thermodynamics as

$$TdS = dE + PdV, \tag{19}$$

where S and E are the total entropy and energy, and V is the volume considered. Taking $\rho = \rho(T)$ and $P = P(T)$ [1], one may write

$$dS(V,T) = \frac{1}{T}[d(\rho V) + PdV]. \tag{20}$$

From Eq. (19) we get

$$\frac{\partial S}{\partial V} = \frac{\rho + P}{T}, \tag{21}$$

$$\frac{\partial S}{\partial T} = \frac{V}{T}\frac{\partial \rho}{\partial T}. \tag{22}$$

Using the integrability condition of these equations

$$\frac{\partial}{\partial T}\left(\frac{\partial S}{\partial V}\right) = \frac{\partial}{\partial V}\left(\frac{\partial S}{\partial T}\right), \tag{23}$$

we end up with

$$\frac{dP}{dT} = \frac{\rho + P}{T}. \tag{24}$$

As in the standard model we assume that the expansion of the universe is adiabatic in all eras. The conservation of energy is now expressed as

$$\frac{d}{dt}(\rho V) = -P\frac{dV}{dt}. \tag{25}$$

Using Eq. (24) we could write

$$\frac{d}{dt}\left[\frac{V}{T}(\rho+P)\right]=0, \tag{26}$$

so that $\frac{V}{T}(\rho+P)$ is a conserved quantity. This is nothing but the total entropy of the system considered in a volume V. This could easily be seen if Eqs. (21) and (22) are taken together with Eq. (24).

In terms of the scale factor, conservation of entropy could be expressed as

$$\frac{d}{dt}\left[\frac{a^3}{T}(\rho+P)\right]=0, \tag{27}$$

which could be written in the form

$$\frac{d}{dt}\left(\frac{a^3\rho}{T}\right)=0, \tag{28}$$

where the general form of the equation of state is used. Carrying out differentiation in Eq. (28), we obtain

$$3\frac{\dot{a}}{a}-\frac{\dot{T}}{T}+\frac{\dot{\rho}}{\rho}=0. \tag{29}$$

Now, we differentiate Eq. (3) and then use Eq. (6) to get the relation

$$\dot{\rho}+3\gamma\rho\frac{\dot{a}}{a}=0, \tag{30}$$

which describes the time evolution of the energy density. Substituting Eq. (30) in Eq. (29) and, as usual, defining the Hubble constant H as

$$H\equiv\frac{\dot{a}}{a}, \tag{31}$$

we could write Eq. (29) in the form

$$\frac{\dot{T}}{T}+3H(\gamma-1)=0. \tag{32}$$

In terms of the conformal time η this becomes

$$\frac{T'}{T}+3\frac{a'}{a}(\gamma-1)=0. \tag{33}$$

This gives us the equation describing the time evolution of the temperature.

B. Boundary Conditions and the Solutions For the Scale Factor

Initially, we assume that the universe is in a vacuum-like state and has a limiting density called the Planck density (ρ_{pl}), which is first formulated by Markov as a universal law of nature [7]. However, we have some difficulties in imposing the initial value of the expansion rate $(a'(0))$. Since the energy density in the universe is positive, Eq. (9) does not allow the initially static condition i.e.,

$$a'(0) = 0, \tag{34}$$

which is used in the singularity-free closed cosmological models as a simplifying assumption [4,5,6]. Since our model describes a universe starting from a finite size and density, the initial expansion rate must be taken as positive. Hence we take the initial expansion rate as

$$a'(0) = v, \tag{35}$$

where v is some positive constant.

In the light of these, we could go back to the field Eq. (9) and write it at $\eta = 0$ as

$$da^4(0) + a^2(0) - v^2 = 0, \tag{36}$$

where $d = \frac{8\pi}{3}\rho_{pl}$. Eq. (36) is quadratic in $a^2(0)$ and has the following physical solution

$$a(0) = \sqrt{\frac{\sqrt{1+4dv^2}-1}{2d}}, \tag{37}$$

which reflects the singularity-free character of our cosmological model. Solutions for the scale factor in different eras are:

$$a(\eta) = \begin{cases} a_0^{(p)}[\sinh(c_p\eta + \delta_p)\,]^{1/c_p} & 0 \leq \eta \leq \eta_r, \\ a_0^{(r)}[\sinh(\eta + \delta_r)] & \eta_r \leq \eta \leq \eta_m, \\ a_0^{(m)}[\sinh(\eta/2 + \delta_m)]^2 & \eta_m \leq \eta. \end{cases} \tag{38}$$

We next impose the boundary condition that the scale factor and its derivative are continuous at points η_r and η_m, where the phase transitions take place. This determines the integration constants as,

$$a_0^{(p)} = \left[\frac{\sqrt{v^2 - a^2(0)}}{a(0)} \right]^{\frac{1}{c_p}+1} d^{-1/2}, \tag{39}$$

$$\delta_p = \ln \sqrt{\frac{v + a(0)}{v - a(0)}}, \tag{40}$$

$$a_0^{(r)} = a_0^{(p)} [\sinh(c_p \eta_r + \delta_p)]^{\frac{1}{c_p}-1}, \tag{41}$$

$$\delta_r = (c_p - 1)\, \eta_r + \delta_p, \tag{42}$$

$$a_0^{(m)} = \frac{a_0^{(r)}}{\sinh\left(\eta_m + \delta_r\right)} \tag{43}$$

$$\delta_m = \eta_m/2 + \delta_r. \tag{44}$$

Instead of keeping v as a parameter, we prefer to work with the initial value of the Hubble constant $(H(0))$ which is defined as

$$H(0) = \frac{a'(0)}{a^2(0)}. \tag{45}$$

In terms of $H(0)$ Eqs. (39) and (40) could be rewritten as

$$a_0^{(p)} = \left[\frac{d}{H^2(0) - d} \right]^{\frac{c_p+1}{2c_p}} d^{-1/2} \;\;, \tag{46}$$

and

$$\delta_p = \ln \sqrt{\frac{H(0) + \sqrt{H^2(0) - d}}{H(0) - \sqrt{H^2(0) - d}}}. \tag{47}$$

We have represented the solution for the scale factor in different eras in terms of the initial value of the Hubble constant in the hope of connecting it to its present value H_0. Therefore, we could have an idea about the value of H_0 rather than fine tuning without explanation to an accuracy of one part in 10^{55} as in the standard model [2].

As seen from the solutions for the scale factor, the model that we propose is a two-parameter model. The parameters are the c_p value and the $H\,(0)$. The former determines the amount of inflation that the universe has experienced during the prematter era and the latter is related to the initial expansion rate of the universe.

C. Physical Quantities in the Model

As mentioned before, the prematter era ends up with a first order phase transition which is followed by a radiation dominated period called the radiation era. Once the universe enters the radiation era, it becomes completely filled with isotropic radiation so that its energy-momentum tensor could be approximated by that of the blackbody where the energy density is given as

$$\rho_{blackbody} = \frac{8\pi^5 (kT_{pl})^4}{15}, \tag{48}$$

where $k = 1.38 \cdot 10^{-16} K^{-1}$ is the Boltzmann constant.

Since the evolution of the energy density is governed by the field Eq. (9) and the temperature at the end of the inflationary era is assumed to be T_{pl}, we could derive the conformal time η_r marking the duration of the prematter era. For this purpose, we consider Eq. (9) together with Eqs. (38), (39), (40), (48) and end up with

$$(v + a(0))e^{2c_p\eta_r} - (v - a\,(0)) = 2e^{c_p\eta_r}\,(1.5201)^{\frac{c_p}{2+2c_p}}\,a(0), \tag{49}$$

where we have restored the numerical value of the velocity of light and the Planck constant.

To solve this equation for η_r we first let

$$x = e^{c_p\eta_r}, \tag{50}$$

and solve Eq. (49) for x. The corresponding roots are

$$x_{1,2} = \frac{(1.5201)^{\frac{c_p}{2+2c_p}} \pm \sqrt{(1.5201)^{\frac{c_p}{c_p+1}} + \left[\left(\frac{v}{a(0)}\right)^2 - 1\right]}}{\frac{v}{a(0)} + 1}. \tag{51}$$

Since the quantity inside the square bracket in Eq. (51) is always positive from Eq. (9), we choose x_1 as the solution and write η_r as

$$\eta_r = \ln\left[\frac{(1.5201)^{\frac{c_p}{2+2c_p}} + \sqrt{(1.5201)^{\frac{c_p}{c_p+1}} + \frac{d}{H^2(0)-d}}}{\frac{H(0)+\sqrt{H^2(0)-d}}{\sqrt{H^2(0)-d}}}\right]^{1/c_p}, \tag{52}$$

where we have substituted v in terms of $H(0)$.

We can now derive an expression corresponding to the conformal time η_m marking the duration of the radiation era. To this end, we make use of Eq. (33) which describes the evolution of the temperature T in each era, where γ is specified accordingly. Knowing that γ equals 4/3 during the radiation era and choosing the integration limits to be the conformal times at which phase transitions occur, one gets from Eq. (33)

$$\frac{a(\eta_r)}{a(\eta_m)} = \frac{T_m}{T_{pl}}, \tag{53}$$

where T_m is the temperature at the last phase transition. Roughly speaking, this temperature corresponds to the decoupling of radiation from matter. Weinberg [1] predicts a value between 10^5K and 10^3K and Kolb and Turner [21] give

$$T_m = T_{now} \cdot 2.32 \cdot 10^4\, \Omega_0 h^2 K, \tag{54}$$

where $h \equiv H_0/(100\ Km \cdot s^{-1} \cdot Mpc^{-1})$ and $T_{now} = 2.7\ K$. Upon this, BCF [5] use the numerical value $\Omega_0 h^2 = 0.396$ giving $T_m = 2.4805 \cdot 10^4\ K$ which lies in the transition range $\sim 10^5K - 10^3K$ given by Weinberg. In this work, we shall use this T_m as the temperature at the end of the radiation era. Then Eq. (53) gives

$$\frac{\sinh(\eta_m + \delta_r)}{\sinh(\eta_r + \delta_r)} = \frac{T_{pl}}{T_m}, \tag{55}$$

which could be written in the form

$$zy^2 - b(z^2 - 1)y - z = 0, \tag{56}$$

where $y = e^{\eta_m + \delta_r}$, $z = e^{\eta_r + \delta_r}$ and $b = \dfrac{T_{pl}}{T_m}$. Eq. (56) is quadratic in y and has the following roots

$$y_{1,2} = \frac{b(z^2 - 1) \pm \sqrt{b^2(z^2 - 1)^2 + 4z^2}}{2z}. \tag{57}$$

Using the fact that $e^{\eta_m + \delta_r} > 0$, we could then write the following expression for η_m

$$\eta_m = \ln\left\{(T_{pl}/2T_m)\left[e^{2(c_p\eta_r + \delta_p)} - 1\right] + \sqrt{(T_{pl}/2T_m)^2\left[e^{2(c_p\eta_r + \delta_p)} - 1\right]^2 + e^{2(c_p\eta_r + \delta_p)}}\right\} \tag{58}$$

$$-\left[(2c_p - 1)\eta_r + 2\delta_p\right].$$

At this point, it is to be noted that after the universe enters the matter era, radiation decouples from matter and determines the temperature of the universe as a perfect fluid with the same equation of state as that of the isotropic radiation. Then equation (33) gives

$$\frac{a\left(\eta_{m}\right)}{a\left(\eta_{now}\right)}=\frac{T_{now}}{T_{m}}, \tag{59}$$

where we have chosen integration limits to be the conformal times corresponding to the second phase transition and the present time. Using Eq. (44), one obtains from Eq. (59)

$$\sinh\left(\eta_{now}/2+\delta_{m}\right)=\sqrt{\frac{T_{m}}{T_{now}}}\sinh\left(\eta_{m}+\delta_{r}\right). \tag{60}$$

To solve this equation for η_{now} we let

$$r=e^{\frac{\eta_{now}}{2}+\delta_{m}}, \tag{61}$$

and

$$q=2\sqrt{\frac{T_{m}}{T_{now}}}\sinh\left(\eta_{m}+\delta_{r}\right). \tag{62}$$

With the above substitutions, we end up with the following quadratic equation

$$r^{2}-qr-1=0, \tag{63}$$

which has the roots

$$r_{1,2}=\frac{q\pm\sqrt{q^{2}+4}}{2}. \tag{64}$$

Since $e^{\frac{\eta_{now}}{2}+\delta_{m}}>0$, one may write the following expression for η_{now}

$$\eta_{now}=2\left\{\ln\left[\sqrt{\frac{T_{m}}{T_{now}}}\sinh\left(\eta_{m}+\delta_{r}\right)+\sqrt{\frac{T_{m}}{T_{now}}\sinh^{2}\left(\eta_{m}+\delta_{r}\right)+1}\right]-\delta_{m}\right\}. \tag{65}$$

We can find the comoving times corresponding to the conformal times η_r, η_m, and η_{now} by using the definition given in Eq. (8). Assuming that $t=0$ at $\eta=0$ we get from Eq. (17)

$$t\left(\eta\right)=a_{0}\int_{0}^{\eta}\left[\sinh\left(\eta' c+\delta\right)\right]^{1/c}d\eta'. \tag{66}$$

This integral depends on the values of c and has to be computed numerically in the prematter era. Whereas, for the radiation ($c_r = 1$) and matter ($c_m = 1/2$) eras, Eq. (66) could be integrated to yield analytical expressions. The expressions for the comoving times corresponding to η_r, η_m and η_{now} are

$$t_r = a_0^{(p)} \int_0^{\eta_r} \left[\sinh\left(c_p\eta + \delta_p\right)\right]^{1/c_p} d\eta, \tag{67}$$

$$t_m = t_r + a_0^{(r)} \left[\cosh\left(\eta_m + \delta_r\right) - \cosh\left(\eta_r + \delta_r\right)\right], \tag{68}$$

$$t_{now} = t_m + \frac{a_0^{(m)}}{2} \left[\sinh\left(\eta_{now} + 2\delta_m\right) - \sinh\left(\eta_m + 2\delta_m\right) + \left(\eta_m - \eta_{now}\right)\right]. \tag{69}$$

Hubble constants at η_r, η_m and η_{now} could now be obtained by using

$$H(\eta) = \frac{a'(\eta)}{a^2(\eta)}, \tag{70}$$

as

$$H\left(\eta_r\right) = \frac{a'\left(\eta_r\right)}{a^2\left(\eta_r\right)} = \frac{\left(9.2503 \cdot 10^{29}\right)\cosh\left(c_p\eta_r + \delta_p\right)}{a_0^{(p)}\left[\sinh\left(c_p\eta_r + \delta_p\right)\right]^{\frac{1}{c_p}+1}} \quad Km \cdot s^{-1} \cdot Mpc^{-1}, \tag{71}$$

$$H\left(\eta_m\right) = \frac{a'\left(\eta_m\right)}{a^2\left(\eta_m\right)} = \frac{\left(9.2503 \cdot 10^{29}\right)\cosh\left(\eta_m + \delta_r\right)}{a_0^{(r)}\sinh^2\left(\eta_m + \delta_r\right)} \quad Km \cdot s^{-1} \cdot Mpc^{-1}, \tag{72}$$

$$H\left(\eta_{now}\right) = \frac{a'\left(\eta_{now}\right)}{a^2\left(\eta_{now}\right)} = \frac{\left(9.2503 \cdot 10^{29}\right)\cosh\left(\eta_{now}/2 + \delta_m\right)}{a_0^{(m)}\sinh^3\left(\eta_{now}/2 + \delta_m\right)} \quad Km \cdot s^{-1} \cdot Mpc^{-1}. \tag{73}$$

The evolution of the energy density is described by Eq. (30), which may also be written as

$$\rho' + 3\gamma\frac{a'}{a}\rho = 0. \tag{74}$$

Eq. (74) could then be solved to yield

$$\frac{\rho\left(\eta_f\right)}{\rho\left(\eta_i\right)} = \left(\frac{a\left(\eta_i\right)}{a\left(\eta_f\right)}\right)^{3\gamma}, \tag{75}$$

where η_i and η_f are the initial and final instants respectively of any conformal time interval in a given era. Choosing $(\eta_{i,}\eta_f) = (0, \eta_r), (\eta_r, \eta_m), (\eta_m, \eta_{now})$ we obtain,respectively

$$\rho(\eta_r) = \left(\frac{a(0)}{a(\eta_r)}\right)^{3\gamma_p} \rho_{pl}, \tag{76}$$

$$\rho(\eta_m) = \left(\frac{a(\eta_r)}{a(\eta_m)}\right)^{4} \rho(\eta_r), \tag{77}$$

$$\rho(\eta_{now}) = \left(\frac{a(\eta_m)}{a(\eta_{now})}\right)^{3} \rho(\eta_m). \tag{78}$$

D. Numerical Solutions For the Physical Quantities in the Model

Even though we have listed all the physical quantities in the previous section, further approximations have to be made in order to construct numerical models. We notice that Eq. (9) puts a restriction on the value of $H(0)$ which could be written as

$$H^2(0) > d, \tag{79}$$

where $d = \dfrac{8\pi}{3}\rho_{pl}$.

Eqs. (47) and (52) are combined to yield

$$c_p\eta_r + \delta_p = \ln\left[\frac{(1.5201)^{\frac{c_p}{2+2c_p}} + \sqrt{(1.5201)^{\frac{c_p}{1+c_p}} + \dfrac{d}{H^2(0)-d}}}{\sqrt{\dfrac{d}{H^2(0)-d}}}\right]. \tag{80}$$

For the sake of simplicity we define a positive parameter

$$\lambda = \frac{(1.5201)^{\frac{c_p}{2+2c_p}}}{\sqrt{\dfrac{d}{H^2(0)-d}}} \tag{81}$$

and rewrite Eq. (80) in terms of λ as

$$c_p\eta_r + \delta_p = \ln\left(\lambda + \sqrt{1+\lambda^2}\right). \tag{82}$$

Since c_p is a negative number, from Eq. (38) the scale factor has a singularity at $\eta = \delta_p/|c_p|$. However, comparing this with η_r leads us to the conclusion that inflation stops before this singularity is actually reached, where a first order phase transition carries us into the radiation era.

Substituting Eq. (82) in Eq. (38), and using Eq. (46) the scale factor at η_r becomes

$$a\left(\eta_r\right) = \frac{(1.5201)^{\frac{1}{2+2c_p}}}{\sqrt{H^2(0) - d}}. \tag{83}$$

Returning back to Eq. (55) and using Eqs. (38), (41) and (42) we obtain the scale factor at η_m as

$$a(\eta_m) = \frac{(1.5201)^{\frac{1}{2+2c_p}}}{\sqrt{H^2(0) - d}} \frac{T_{pl}}{T_m}. \tag{84}$$

Going back to Eq. (60) and making use of Eq. (38) and (43) the present value of the scale factor could be written as

$$a\left(\eta_{now}\right) = \frac{(1.5201)^{\frac{1}{2+2c_p}}}{\sqrt{H^2(0) - d}} \frac{T_{pl}}{T_{now}}. \tag{85}$$

We could now obtain expressions for the real times t_m and t_{now}. Using Eq. (68) together with Eqs. (41), (42), (46), (55) and (82) we get t_m as

$$t_m = t_r + \left(\frac{d}{H^2(0) - d}\right)^{\frac{c_p+1}{2c_p}} d^{-1/2} \lambda^{\frac{1-c_p}{c_p}} \left[\sqrt{1 + \left(\lambda T_{pl}/T_m\right)^2} - \sqrt{1+\lambda^2}\right]. \tag{86}$$

We could now evaluate t_{now} by writing

$$\sinh\left(\eta_{now} + 2\delta_m\right) = 2\sinh\left(\eta_{now}/2 + \delta_m\right)\sqrt{1 + \sinh^2\left(\eta_{now}/2 + \delta_m\right)}, \tag{87}$$

and making use of Eqs. (42), (55), (60) and (82) this could be written as

$$\sinh\left(\eta_{now} + 2\delta_m\right) = 2\lambda \frac{T_{pl}}{\sqrt{T_m T_{now}}} \sqrt{1 + \left(\lambda \frac{T_{pl}}{\sqrt{T_m T_{now}}}\right)^2}. \tag{88}$$

In a similar way, using Eq. (44) we may write

$$\sinh\left(\eta_m + 2\delta_m\right) = 2\sinh\left(\eta_m + \delta_r\right)\sqrt{1 + \sinh^2\left(\eta_m + \delta_r\right)}, \tag{89}$$

and making use of Eqs. (42), (55) and (82) one gets

$$\sinh\left(\eta_m + 2\delta_m\right) = 2\lambda \frac{T_{pl}}{T_m}\sqrt{1+\left(\lambda\frac{T_{pl}}{T_m}\right)^2}. \tag{90}$$

We could also find an expression for $\eta_m - \eta_{now}$ by returning back to equation (65) and using Eqs. (42), (44), (55) and (82). That is

$$\eta_m - \eta_{now} = 2\left\{\left(\eta_m + \delta_r\right) - \ln\left[\lambda\frac{T_{pl}}{\sqrt{T_m T_{now}}} + \sqrt{1+\left(\lambda\frac{T_{pl}}{\sqrt{T_m T_{now}}}\right)^2}\right]\right\}. \tag{91}$$

Making use of Eq. (55), the first term in parenthesis could be evaluated as

$$\eta_m + \delta_r = \ln\left[\lambda\frac{T_{pl}}{T_m} + \sqrt{1+\left(\lambda\frac{T_{pl}}{T_m}\right)^2}\right], \tag{92}$$

which, when combined with Eqs. (69), (88), (90), (91) and (92), leads us to write the present age of the universe as

$$\begin{aligned} t_{now} = t_m + a_0^{(m)} &\left\{\lambda\frac{T_{pl}}{\sqrt{T_m T_{now}}}\sqrt{1+\left(\lambda\frac{T_{pl}}{\sqrt{T_m T_{now}}}\right)^2} - \lambda\frac{T_{pl}}{T_m}\sqrt{1+\left(\lambda\frac{T_{pl}}{T_m}\right)^2}\right. \\ &\left. + \ln\left[\frac{\lambda\frac{T_{pl}}{T_m} + \sqrt{1+\left(\lambda\frac{T_{pl}}{T_m}\right)^2}}{\lambda\frac{T_{pl}}{\sqrt{T_m T_{now}}} + \sqrt{1+\left(\lambda\frac{T_{pl}}{\sqrt{T_m T_{now}}}\right)^2}}\right]\right\}, \end{aligned} \tag{93}$$

Returning back to Eq. (71) and making use of Eqs. (46), (81) and (82), we get the Hubble constant at η_r as

$$H\left(\eta_r\right) = \sqrt{1+\lambda^2}\sqrt{\frac{d}{1.5201}}\left(9.2503 \cdot 10^{29}\right)\ Km \cdot s^{-1} \cdot Mpc^{-1}. \tag{94}$$

The Hubble constant at the end of the radiation era takes the following form

$$H(\eta_m) = \sqrt{1+\left(\lambda\frac{T_{pl}}{T_m}\right)^2}\sqrt{\frac{d}{1.5201}}\left(\frac{T_m}{T_{pl}}\right)^2\left(9.2503 \cdot 10^{29}\right)\ Km \cdot s^{-1} \cdot Mpc^{-1}, \tag{95}$$

where Eqs. (41), (42), (55), (72), (81) and (82) have been used.

Finally, making use of Eqs. (41), (42), (43), (46), (55), (73), (81) and (82) the present value of the Hubble constant could be written in the form

$$H(\eta_{now}) = \sqrt{1 + \left(\lambda \frac{T_{pl}}{\sqrt{T_m T_{now}}}\right)^2} \frac{\sqrt{(d/1.5201)\, T_m T_{now}^3}}{T_{pl}^2} (9.2503 \cdot 10^{29}) \; Km \cdot s^{-1} \cdot Mpc^{-1}. \tag{96}$$

To find the regime which gives realistic results, we will use the present value of the Hubble constant as an indicator. We will treat $H(\eta_{now})$ in Eq. (96) for the following two cases separately until we find the realistic regime.

$$For\ case\ 1 : \lambda \geq 1$$

For this case, we could write

$$\sqrt{1 + \left(\lambda \frac{T_{pl}}{\sqrt{T_m T_{now}}}\right)^2} \simeq \lambda \frac{T_{pl}}{\sqrt{T_m T_{now}}}, \tag{97}$$

and using Eq. (96) we get

$$H\left(\eta_{now}\right) \simeq \lambda \sqrt{\frac{d}{1.5201}} \frac{T_{now}}{T_{pl}} (9.2503 \cdot 10^{29}) \; Km \cdot s^{-1} \cdot Mpc^{-1}. \tag{98}$$

After restoring all the numerical values we obtain

$$H\left(\eta_{now}\right) \simeq 2.5603 \cdot 10^{31} \cdot \lambda \; Km \cdot s^{-1} \cdot Mpc^{-1}. \tag{99}$$

From here, we argue that case 1 produces unreasonably high values for the Hubble constant and can not be considered as a realistic regime.

$$For\ case\ 2 : 0 < \lambda < 1$$

For this case we note that Eq. (96) sets a lower limit on the acceptable values of the Hubble constant. This could be seen if one considers the following inequality

$$H\left(\eta_{now}\right) > \left[\sqrt{\frac{d}{1.5201}} \frac{\sqrt{T_m T_{now}^3}}{T_{pl}^2}\right] (9.2503 \cdot 10^{29}) \; Km \cdot s^{-1} \cdot Mpc^{-1}. \tag{100}$$

Restoring all the numerical values, we end up with

$$H\left(\eta_{now}\right) > 46.7634\ Km \cdot s^{-1} \cdot Mpc^{-1}. \tag{101}$$

This lower limit indicates that the values of λ which fall between 0 and 1 could produce realistic results about the present value of the Hubble constant. Since recent observational results for the present value of the Hubble constant lie approximately between 60 and 80 $Km \cdot s^{-1} \cdot Mpc^{-1}$, from Eq. (96) we may write the following requirement for λ

$$1.4683 \cdot 10^{-30} \lesssim \lambda \lesssim 2.5352 \cdot 10^{-30}. \tag{102}$$

The results about the present properties of the universe that follow from case 2 clearly show that our model is sensitive to both parameters ($H(0)$, γ_p). In order to see this dependence and the development of the universe in this model, we change one of the parameters while fixing the other and provide some numerical results. This will allow us to identify the character of the dependency of the model to each parameter. The numerical results correspond to the conformal and comoving times with the corresponding values for the scale factor, energy density, Hubble constant and Ω (ρ/ρ_c) at the transitions between prematter, radiation, matter eras and the present time.

First, we fix $H(0)$ as $\sqrt{2d}$ which is numerically equal to $2.3427 \cdot 10^{63}$ $Km \cdot s^{-1} \cdot Mpc^{-1}$, and change γ_p in such a way that it takes values between $2.0000 \cdot 10^{-3}$ and $2.0500 \cdot 10^{-3}$. It is to be noted that in this broad range, the value of Hubble constant has a wide spectrum between 49.2304 $Km \cdot s^{-1} \cdot Mpc^{-1}$ and 96.5196 $Km \cdot s^{-1} \cdot Mpc^{-1}$. This range corresponds to λ values between $6.0105 \cdot 10^{-31}$ and $3.2978 \cdot 10^{-30}$. The γ_p values between $2.0260 \cdot 10^{-3}$ and $2.0410 \cdot 10^{-3}$ give results which agree with the recent observational data $\left(1.4720 \cdot 10^{-30} \leq \lambda \leq 2.4424 \cdot 10^{-30}\right)$. Results corresponding to the $\gamma_p = 2.0260 \cdot 10^{-3}$, $2.0290 \cdot 10^{-3}$, $2.0320 \cdot 10^{-3}$, $2.0350 \cdot 10^{-3}$, $2.0380 \cdot 10^{-3}$, $2.0410 \cdot 10^{-3}$ are listed in tables 1-6.

Next, we set γ_p to $2.0350 \cdot 10^{-3}$ and vary $H(0)$ in the range between $0.75\sqrt{2d}$ and $1.30\sqrt{2d}$. This range corresponds to λ values between $7.0583 \cdot 10^{-31}$ and $3.0799 \cdot 10^{-30}$. The corresponding values for the Hubble constant varies between 50.1328 $Km \cdot s^{-1} \cdot Mpc^{-1}$ and 91.6791 $Km \cdot s^{-1} \cdot Mpc^{-1}$. $H(0)$ values between $1.1500\sqrt{2d}$ and $0.9000\sqrt{2d}$

$(2.6941 \cdot 10^{63} - 2.1084 \cdot 10^{63} Km \cdot s^{-1} \cdot Mpc^{-1})$ produce results compatible with observations. In tables 7-11 we present the results for $H(0) = 1.1500\sqrt{2d}$, $1.1000\sqrt{2d}$, $1.0500\sqrt{2d}$, $0.9500\sqrt{2d}$, $0.9000\sqrt{2d}$ $(1.5720 \cdot 10^{-30} \leq \lambda \leq 2.5605 \cdot 10^{-30})$.

The energy density value for each era is rather insensitive to both parameters (γ_p and $H(0)$). However, the value of Ω is strongly dependent on these parameters. At the end of the prematter era, the value of Ω approaches to unity which is the Ω value of a universe having a flat space-time geometry ($k = 0$). During the radiation era, Ω remains very close to unity whereas it drops significantly during the matter era and at the present time it takes values compatible with the currently accepted range.

III. CONCLUSION

We construct an open singularity-free cosmological model with the assumptions that the universe is initially in a vacuum like state and the physical quantities are limited by their Planck values. Evolution of the temperature of the universe is governed by its expansion. During the prematter era, temperature increases due to inflation. This era ends when the temperature reaches the maximum allowed temperature T_{pl}. Then the universe enters the radiation era and starts cooling down. This cooling continues during the matter era described by the standard model. The so-called singularity problem is solved by assuming that the universe has initially the Planck density ρ_{pl} which is considered as the natural limiting density. Therefore, the universe has initially a finite scale factor corresponding to this density unlike the standard model in which initial singular behavior of the scale factor causes infinities in physical quantities [3].

In open universe case, unlike the previous closed universe models, we cannot use the initially static condition ($\dot{a}(0) = 0$) since the energy density in the universe has to be positive. Instead of working directly with the initial expansion velocity v, we prefer working with the initial value of the Hubble constant ($H(0)$) which is also a measure of the initial expansion rate of the universe. During the inflationary era, the universe is characterized by a vacuum like equation of state in the form $P = (\gamma_p - 1)\rho$ where $\gamma_p \sim 10^{-3}$ is a parameter which determines the vacuum dominance of the early universe. Hence, the model we construct is a two-parameter universe model, the parameters being $H(0)$ and γ_p.

Without any prior assumption for the initial value of the Hubble constant, we solved for the dynamics of the universe in conformal time instead of the more commonly used comoving time and related the present properties of the universe such as Hubble constant, age and density to the initial value of the Hubble constant. It is worth noting that various values for the parameter γ_p have to be chosen in such a way ($\sim 10^{-3}$) not to disturb the vacuum like characteristic of the early universe. On the other hand, the other parameter $H(0)$ could take values from a broad range determined by the field equations. Since recent observations estimate a value between 60 and 80 $Km \cdot s^{-1} \cdot Mpc^{-1}$ for the Hubble constant H_0, we considered the expression corresponding to the present value of H_0 as an indicator. We analyzed this expression on the basis of possible $H(0)$ and γ_p values and concluded that values in a specific range should be assigned to $H(0)$ as a parameter if we want to get numerical results compatible with the above currently accepted range for H_0. Of course, our model works for the other choices of $H(0)$ but in these cases, it does not produce realistic results for the present properties of the universe (H_o, t_{now}, ρ_{now}, Ω_o) and cannot be used for the description of the universe. This approach is interesting in the sense that it does not set the initial expansion rate of the universe to a specific value unlike other works [4-6] in which the universe is assumed without explanation to expand from an initially static state. Instead, it offers us a way of making predictions about the initial value of the Hubble constant considered as one of the parameters of the model.

Physically significant conformal times have been converted to comoving times and numerical results have been obtained about the properties of the universe (H, t, ρ, Ω) at the end of each era and present time. The values obtained for the present time are in a range compatible with the recent observational results. This clearly shows that the model constructed in this paper could be used as a realistic singularity-free open inflation model.

The so-called flatness problem is not present in this model since the initial value of Ω

is not fine tuned to unity. In the standard model, such a precise initial condition has to be assumed without explanation to produce a universe resembling the actual one. In this model, the universe starts its journey with an Ω value no matter how close to unity and during the prematter era Ω is driven toward unity. At the end of this era, the universe is nearly flat since its Ω value is very close to one. This fact could be attributed to the inflation mechanism which causes the universe to expand enormously in a very small time interval (Planck time) and thus become flatter than at the beginning. During the radiation and matter eras, Ω is driven away unity which apparently displays the open character of the space-time geometry of the model.

In this paper, we have not been interested in the microphysics of the inflationary era. This includes quantum mechanical investigation of the material content of the early universe and the form of the scalar field potential responsible for the inflation. However, with this simple form, the scenario proposed in this work might provide a guidance for the future more complete versions of the open inflation.

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TABLE 1

RESULTS FOR $\gamma_p = 2.0260 \cdot 10^{-3}$ and $H(0) = 2.3427 \cdot 10^{63} Km \cdot s^{-1} \cdot Mpc^{-1}$

η	$t\ (yr)$	$a\ (cm)$	$\rho\ (\frac{gr}{cm^3})$	$H\ \left(\frac{km}{s \cdot mpc}\right)$	Ω
0	0	$5.5841 \cdot 10^{-34}$	$5.1566 \cdot 10^{93}$	$2.3427 \cdot 10^{63}$	0.5000
0.8841	*very small #*	$4.6771 \cdot 10^{-4}$	$3.3923 \cdot 10^{93}$	$1.3436 \cdot 10^{63}$	$1 - 2.1668 \cdot 10^{-60}$
0.8925	$1.1872 \cdot 10^{4}$	$2.6716 \cdot 10^{24}$	$3.1863 \cdot 10^{-18}$	$4.1180 \cdot 10^{7}$	0.9999
2.3502	$1.1895 \cdot 10^{10}$	$2.4544 \cdot 10^{28}$	$4.1093 \cdot 10^{-30}$	60.0600	0.6062

TABLE 2

RESULTS FOR $\gamma_p = 2.0290 \cdot 10^{-3}$ and $H(0) = 2.3427 \cdot 10^{63} Km \cdot s^{-1} \cdot Mpc^{-1}$

η	$t(yr)$	$a(cm)$	$\rho\ (\frac{gr}{cm^3})$	$H\ \left(\frac{km}{s \cdot mpc}\right)$	Ω
0	0	$5.5841 \cdot 10^{-34}$	$5.1566 \cdot 10^{93}$	$2.3427 \cdot 10^{63}$	0.5000
0.8841	*very small #*	$4.2241 \cdot 10^{-4}$	$3.3923 \cdot 10^{93}$	$1.3436 \cdot 10^{63}$	$1 - 2.6565 \cdot 10^{-60}$
0.8934	$1.1872 \cdot 10^{4}$	$2.4129 \cdot 10^{24}$	$3.1863 \cdot 10^{-18}$	$4.1180 \cdot 10^{7}$	0.9999
2.4811	$1.1558 \cdot 10^{10}$	$2.2167 \cdot 10^{28}$	$4.1093 \cdot 10^{-30}$	62.6753	0.5567

TABLE 3

RESULTS FOR $\gamma_p = 2.0320 \cdot 10^{-3}$ and $H(0) = 2.3427 \cdot 10^{63} Km \cdot s^{-1} \cdot Mpc^{-1}$

η	$t\ (yr)$	$a\ (cm)$	$\rho\ (\frac{gr}{cm^3})$	$H\ \left(\frac{km}{s \cdot mpc}\right)$	Ω
0	0	$5.5841 \cdot 10^{-34}$	$5.1566 \cdot 10^{93}$	$2.3427 \cdot 10^{63}$	0.5000
0.8841	*very small #*	$3.8161 \cdot 10^{-4}$	$3.3923 \cdot 10^{93}$	$1.3436 \cdot 10^{63}$	$1 - 3.2548 \cdot 10^{-60}$
0.8944	$1.1872 \cdot 10^{4}$	$2.1798 \cdot 10^{24}$	$3.1863 \cdot 10^{-18}$	$4.1181 \cdot 10^{7}$	0.9999
2.6191	$1.1188 \cdot 10^{10}$	$2.0026 \cdot 10^{28}$	$4.1093 \cdot 10^{-30}$	65.7300	0.5062

TABLE 4

RESULTS FOR $\gamma_p = 2.0350 \cdot 10^{-3}$ and $H(0) = 2.3427 \cdot 10^{63} Km \cdot s^{-1} \cdot Mpc^{-1}$

η	$t\ (yr)$	$a\ (cm)$	$\rho\ (\frac{gr}{cm^3})$	$H\ \left(\frac{km}{s \cdot mpc}\right)$	Ω
0	0	$5.5841 \cdot 10^{-34}$	$5.1566 \cdot 10^{93}$	$2.3427 \cdot 10^{63}$	0.5000
0.8841	*very small #*	$3.4486 \cdot 10^{-4}$	$3.3923 \cdot 10^{93}$	$1.3436 \cdot 10^{63}$	$1 - 3.9856 \cdot 10^{-60}$
0.8955	$1.1872 \cdot 10^{4}$	$1.9699 \cdot 10^{24}$	$3.1863 \cdot 10^{-18}$	$4.1181 \cdot 10^{7}$	0.9999
2.7640	$1.0786 \cdot 10^{10}$	$1.8097 \cdot 10^{28}$	$4.1093 \cdot 10^{-30}$	69.2782	0.4556

TABLE 5

RESULTS FOR $\gamma_p = 2.0380 \cdot 10^{-3}$ and $H(0) = 2.3427 \cdot 10^{63} Km \cdot s^{-1} \cdot Mpc^{-1}$

η	$t(yr)$	$a(cm)$	$\rho\ (\frac{gr}{cm^3})$	$H\ \left(\frac{km}{s \cdot mpc}\right)$	Ω
0	0	$5.5841 \cdot 10^{-34}$	$5.1566 \cdot 10^{93}$	$2.3427 \cdot 10^{63}$	0.5000
0.8841	*very small #*	$3.1174 \cdot 10^{-4}$	$3.3923 \cdot 10^{93}$	$1.3436 \cdot 10^{63}$	$1 - 4.8775 \cdot 10^{-60}$
0.8967	$1.1872 \cdot 10^{4}$	$1.7807 \cdot 10^{24}$	$3.1863 \cdot 10^{-18}$	$4.1182 \cdot 10^{7}$	0.9998
2.9151	$1.0354 \cdot 10^{10}$	$1.6359 \cdot 10^{28}$	$4.1093 \cdot 10^{-30}$	73.3767	0.4062

TABLE 6

RESULTS FOR $\gamma_p = 2.0410 \cdot 10^{-3}$ and $H(0) = 2.3427 \cdot 10^{63} Km \cdot s^{-1} \cdot Mpc^{-1}$

η	$t(yr)$	$a(cm)$	$\rho\ (\frac{gr}{cm^3})$	$H\ \left(\frac{km}{s \cdot mpc}\right)$	Ω
0	0	$5.5841 \cdot 10^{-34}$	$5.1566 \cdot 10^{93}$	$2.3427 \cdot 10^{63}$	0.5000
0.8841	*very small #*	$2.8188 \cdot 10^{-4}$	$3.3923 \cdot 10^{93}$	$1.3436 \cdot 10^{63}$	$1 - 5.9655 \cdot 10^{-60}$
0.8980	$1.1872 \cdot 10^{4}$	$1.6101 \cdot 10^{24}$	$3.1863 \cdot 10^{-18}$	$4.1182 \cdot 10^{7}$	0.9998
3.0720	$9.8975 \cdot 10^{9}$	$1.4792 \cdot 10^{28}$	$4.1093 \cdot 10^{-30}$	78.0854	0.3587

TABLE 7

RESULTS FOR $H(0) = 2.6941 \cdot 10^{63} Km \cdot s^{-1} \cdot Mpc^{-1}$ and $\gamma_p = 2.0350 \cdot 10^{-3}$

η	$t(yr)$	$a(cm)$	$\rho\ (\frac{gr}{cm^3})$	$H\ \left(\frac{km}{s \cdot mpc}\right)$	Ω
0	0	$4.3538 \cdot 10^{-34}$	$5.1566 \cdot 10^{93}$	$2.6941 \cdot 10^{63}$	0.3781
1.0711	*very small #*	$2.6888 \cdot 10^{-4}$	$3.3923 \cdot 10^{93}$	$1.3436 \cdot 10^{63}$	$1 - 6.5563 \cdot 10^{-60}$
1.0857	$1.1871 \cdot 10^4$	$1.5359 \cdot 10^{24}$	$3.1863 \cdot 10^{-18}$	$4.1183 \cdot 10^7$	0.9998
3.3346	$9.6756 \cdot 10^9$	$1.4110 \cdot 10^{28}$	$4.1093 \cdot 10^{-30}$	80.5275	0.3372

TABLE 8

RESULTS FOR $H(0) = 2.5770 \cdot 10^{63} Km \cdot s^{-1} \cdot Mpc^{-1}$ and $\gamma_p = 2.0350 \cdot 10^{-3}$

η	$t(yr)$	$a(cm)$	$\rho\ (\frac{gr}{cm^3})$	$H\ \left(\frac{km}{s \cdot mpc}\right)$	Ω
0	0	$4.6860 \cdot 10^{-34}$	$5.1566 \cdot 10^{93}$	$2.5770 \cdot 10^{63}$	0.4132
1.0137	*very small #*	$2.8940 \cdot 10^{-4}$	$3.3923 \cdot 10^{93}$	$1.3436 \cdot 10^{63}$	$1 - 5.6595 \cdot 10^{-60}$
1.0273	$1.1872 \cdot 10^4$	$1.6531 \cdot 10^{24}$	$3.1863 \cdot 10^{-18}$	$4.1182 \cdot 10^7$	0.9998
3.1601	$1.0019 \cdot 10^{10}$	$1.5187 \cdot 10^{28}$	$4.1093 \cdot 10^{-30}$	76.7907	0.3708

TABLE 9

RESULTS FOR $H(0) = 2.4599 \cdot 10^{63} Km \cdot s^{-1} \cdot Mpc^{-1}$ and $\gamma_p = 2.0350 \cdot 10^{-3}$

η	$t(yr)$	$a(cm)$	$\rho\ (\frac{gr}{cm^3})$	$H\ \left(\frac{km}{s \cdot mpc}\right)$	Ω
0	0	$5.0869 \cdot 10^{-34}$	$5.1566 \cdot 10^{93}$	$2.4599 \cdot 10^{63}$	0.4535
0.9517	*very small #*	$3.1416 \cdot 10^{-4}$	$3.3923 \cdot 10^{93}$	$1.3436 \cdot 10^{63}$	$1 - 4.8026 \cdot 10^{-60}$
0.9642	$1.1872 \cdot 10^4$	$1.7945 \cdot 10^{24}$	$3.1863 \cdot 10^{-18}$	$4.1182 \cdot 10^7$	0.9998
2.9710	$1.0388 \cdot 10^{10}$	$1.6486 \cdot 10^{28}$	$4.1093 \cdot 10^{-30}$	73.0416	0.4099

TABLE 10

RESULTS FOR $H(0) = 2.2256 \cdot 10^{63} Km \cdot s^{-1} \cdot Mpc^{-1}$ and $\gamma_p = 2.0350 \cdot 10^{-3}$

η	$t(yr)$	$a(cm)$	$\rho\ (\frac{gr}{cm^3})$	$H\ \left(\frac{km}{s \cdot mpc}\right)$	Ω
0	0	$6.2238 \cdot 10^{-34}$	$5.1566 \cdot 10^{93}$	$2.2256 \cdot 10^{63}$	0.5540
0.8093	*very small #*	$3.8436 \cdot 10^{-4}$	$3.3923 \cdot 10^{93}$	$1.3436 \cdot 10^{63}$	$1 - 3.2084 \cdot 10^{-60}$
0.8195	$1.1872 \cdot 10^{4}$	$2.1955 \cdot 10^{24}$	$3.1863 \cdot 10^{-18}$	$4.1180 \cdot 10^{7}$	0.9999
2.5343	$1.1215 \cdot 10^{10}$	$2.0171 \cdot 10^{28}$	$4.1093 \cdot 10^{-30}$	65.4979	0.5097

TABLE 11

RESULTS FOR $H(0) = 2.1084 \cdot 10^{63} Km \cdot s^{-1} \cdot Mpc^{-1}$ and $\gamma_p = 2.0350 \cdot 10^{-3}$

η	$t(yr)$	$a(cm)$	$\rho\ (\frac{gr}{cm^3})$	$H\ \left(\frac{km}{s \cdot mpc}\right)$	Ω
0	0	$7.0918 \cdot 10^{-34}$	$5.1566 \cdot 10^{93}$	$2.1084 \cdot 10^{63}$	0.6173
0.7250	*very small #*	$4.3797 \cdot 10^{-4}$	$3.3923 \cdot 10^{93}$	$1.3436 \cdot 10^{63}$	$1 - 2.4711 \cdot 10^{-60}$
0.7340	$1.1872 \cdot 10^{4}$	$2.5018 \cdot 10^{24}$	$3.1863 \cdot 10^{-18}$	$4.1180 \cdot 10^{7}$	0.9999
2.2747	$1.1682 \cdot 10^{10}$	$2.2984 \cdot 10^{28}$	$4.1093 \cdot 10^{-30}$	61.6978	0.5745